MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2014/15
– Blatt 8 –
10.12.2014

PDG I
(Zentralübung)

Problem Sheet 9

Question 1

(a) Prove Theorem 47 from the lectures: suppose $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$. Define

$$u(t,x) := \tfrac{1}{2}\big[g(x+t) + g(x-t)\big] + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}\,.$$

Then

(i) $u \in C^2([0,\infty) \times \mathbb{R})$.

(ii) $u_{tt} - u_{xx} = 0$ in $(0,\infty) \times \mathbb{R}$.

(iii)

$$\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = g(x_0)\,, \qquad \lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = h(x_0) \quad \text{for all } x_0 \in \mathbb{R}\,.$$

(b) Now let $f \in C^1((0,\infty) \times \mathbb{R})$. Prove that the function $v(t,x)$, defined (as in lectures) as

$$v(t,x) := \tfrac{1}{2}\big[g(x+t) + g(x-t)\big] + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y + \tfrac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,\mathrm{d}y\,\mathrm{d}s\,,$$

satisfies $v_{tt} - v_{xx} = f$ in $(0,\infty) \times \mathbb{R}$ (i.e. v solves the inhomogenous wave equation in one dimension).

Question 2

Let $f \in C^2((0,\infty) \times \mathbb{R})$, $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and consider the initial-value problem for the one-dimensional wave equation:

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } (0,\infty) \times \mathbb{R} \\ u = g \quad u_t = h & \text{on } \{t = 0\} \times \mathbb{R}\,. \end{cases}$$

(a) Show that the solution for $f = g = 0$ is given by

$$u(t,x) = \int_{\mathbb{R}} K(t, x-y) h(y)\,\mathrm{d}y\,,$$

where $K(t,x) := \frac{1}{2}H(|t| - |x|)\mathrm{sgn}(t)$, and H is the characteristic function of the interval $[0,\infty)$ (the "Heaviside function").

Hint: Use d'Alembert's formula.

(b) Use Duhamel's principle to determine the solution for $g = h = 0$, f non-zero.

Question 3

Prove that the general solution $u \in C^2(\mathbb{R}^2)$ of

$$u_{xy}(x,y) = 0\,, \quad (x,y) \in \mathbb{R}^2$$

is given by $u(x,y) = \xi(x) + \zeta(y)$, where $\xi, \zeta \in C^2(\mathbb{R})$.

Question 4

Let u be the solution of the oscillating string problem on $\mathbb{R}_+$, as covered in lectures for $g \in C^2(\overline{\mathbb{R}_+})$, $h \in C^1(\overline{\mathbb{R}_+})$. Prove that if $g''(0) = g(0) = h(0) = 0$, then $u \in C^2([0,\infty) \times \overline{\mathbb{R}_+})$.

(Recall $\mathbb{R}_+ := (0,\infty)$, $\overline{\mathbb{R}_+} = [0,\infty)$.)

Deadline for handing in: 0800 Wednesday 17 December

Please put solutions in Box 17, 1st floor (near the library)

Problem Sheet 9

① Satz 47: $g \in C^2(\mathbb{R})$ $h \in C^1(\mathbb{R})$

$$u(t,x) := \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \qquad (t,x) \in (0,\infty)\times\mathbb{R}$$

(i) $u \in C^2([0,\infty)\times\mathbb{R})$

(ii) $u_{tt} - u_{xx} = 0$ in $(0,\infty)\times\mathbb{R}$

(iii) $\lim\limits_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = g(x_0)$ $\qquad \lim\limits_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = h(x_0)$

Pf: $u_{\cancel{t}}(t,x) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_0^{x+t} h(y)\,dy - \frac{1}{2}\int_0^{x-t} h(y)\,dy$

$u_x(t,x) = \frac{1}{2}\left[g'(x+t) + g'(x-t)\right] + \frac{1}{2}h(x+t) - \frac{1}{2}h(x-t)$ (FTC) $\in C$

$u_{xx}(t,x) = \frac{1}{2}\left[g''(x+t) + g''(x-t)\right] + \frac{1}{2}\left[h'(x+t) - \cancel{*}\, h'(x-t)\right)$

def of u makes sense

$g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ so $u \in C(\mathbb{R}\times\mathbb{R})$ (even if $t<0$)

$u_x \in C(\mathbb{R}\times\mathbb{R})$

$u_{xx} \in C(\mathbb{R}\times\mathbb{R})$

$u_{\cancel{x}t}(t,x) = \frac{1}{2}\left[g'(x+t) - g'(x-t)\right] + \frac{1}{2}h(x+t) \pm$

$u_{tt}(t,x) = \frac{1}{2}\left[g''(x+t) + g''(x-t)\right] + \frac{1}{2}\left[h'(x+t) - h'(x-t)\right]$

$= u_{xx}(t,x) \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}.$

$u_{tx} = \frac{1}{2}\left[g''(x+t) - g''(x-t)\right] + \frac{1}{2}\left[h'(x+t) + h'(x-t)\right] \in C(\mathbb{R}\times\mathbb{R})$

so $u \in C^2(\mathbb{R}\times\mathbb{R})$ (hence $u \in C^2([0,\infty)\times\mathbb{R})$)

Since $u \in C^2(\mathbb{R}\times\mathbb{R})$,

$\lim\limits_{(t,x)\to(0,x_0)} u(t,x) = u(0,x_0) = g(x_0)$

Sim $\lim u_t(t,x) = \frac{1}{2}\left[g'(x_0) - g'(x_0)\right] + \frac{1}{2}\left[h(x_0) + h(x_0)\right]$

$= h(x_0)$

$\square$

(b) Now let $f \in C^1((0,\infty)\times\mathbb{R})$. Define

$$v(t,x) := \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dy\,ds.$$

Claim: $v_{tt} - v_{xx} = f$ in $(0,\infty)\times\mathbb{R}$.

Pf: ~~By (a)~~ let $w(t,x) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dy\,ds$

By (a), suffices to show $w_{tt} - w_{xx} = f$ in $(0,\infty)\times\mathbb{R}$.

$$w_x(t,x) = \frac{1}{2}\int_0^t \underbrace{\left(\frac{\partial}{\partial x}\int_{x-t+s}^{x+t-s} f(s,y)\,dy\right)} ds$$

$$\frac{1}{2}\int_0^t \left(\frac{\partial}{\partial x}\left(\int_0^{x+t-s} f(s,y)\,dy - \int_0^{x-t+s} f(s,y)\,dy\right)\right) ds$$

$$= \frac{1}{2}\int_0^t f(s,x+t-s) - f(s,x-t+s)\,ds$$

$$w_{xx}(t,x) = \frac{1}{2}\int_0^t \cancel{\partial_2}\ \frac{\partial}{\partial x} f(s,x+t-s) - \frac{\partial}{\partial x} f(s,x-t+s)\,ds$$

$$= \frac{1}{2}\int_0^t \partial_2 f(s,x+t-s) - \partial_2 f(s,x-t+s)\,ds$$

derivative wrt 2nd variable of f
$\left(\lim_{h\to 0}\ \ \cancel{f(...)}\ \ \frac{\partial}{\partial y_2} f(y_1,y_2)\right)$

~~$w_t(t,x) =$ Recall~~
~~$\frac{d}{dt}\int_0^t F(x,t)\,dx = F(t,t) - \int_0^t \partial_2 F(x,t)\,dx$~~

~~Write~~ $F(s,$

Recall $\frac{d}{dt}\int_0^t F(s,t)\,ds = F(t,t) + \int_0^t \partial_2 F(s,t)\,dt$

Write $F(s,t) := \int_{x-t+s}^{x+t-s} f(s,y)\,dy \qquad (s,t)\in\mathbb{R}^2$

Thoz $w_t(t,x) = \frac{d}{dt} \frac{1}{2}\int_0^t F(s,t)\, ds$

$$= \frac{1}{2}\left[F(t,t) + \cancel{\tfrac{1}{2}} \int_0^t \partial_2 F_t(s,t)\, ds\right]$$

$$w_{tt}(t,x) = \frac{1}{2}\left[\underbrace{\frac{d}{dt}F(t,t)}_{=0} + \frac{d}{dt}\int_0^t F_t(s,t)\,ds\right]$$

$$\left[F(t,t) = \int_{x-t+s}^{x+t-s} \cancel{F(s,y)}\, f(t,y)\, dy\right]$$

$$\frac{d}{dt}F(t,t) = \partial_1 F(t,t) + \partial_2 F(t,t)$$

$$\partial_1 F(s,t) = \frac{\partial}{\partial s}\left(\int_0^{x+t-s} f(s,y)\,dy \cancel{+} - \int_0^{x-t+s} f(s,y)\,dy\right)$$

$$= -f(\cancel{x+t+s}\, s, x+t-s) - f(s, x-t+s)$$
$$+ \int_{x-t+s}^{x+t-s} \partial_1 f(s,y)\,dy$$

(1) $\partial F(s,t) = \frac{\partial}{\partial t}\int_{x-t+s}^{x+t-s} f(s,y)\,dy = f(s,x+t-s) + f(s,x-t+s)$

So $\frac{d}{dt}F(t,t) = -f(s,x+t-t) - f(s,x-t+t) + f(s,x+t-t)$
$+ f(s,x-t+t) + \int_x^x \partial_1 f(t,y)\,dy$

$= 0.$

$$\frac{d}{dt}\int_0^t F_t(s,t)\,ds = \partial_2 F_{\cancel{tt}}(t,t) + \int_0^t \partial_2^2 F(s,t)\,ds$$

by (1) $\partial_2 F(t,t) = \cancel{2}\; 2f(t,x)$

$$\partial_2^2 F(s,t) = \cancel{\partial_2 (f}\frac{d}{dt}\left(f(s,x+t-s) + f(s,x-t+s)\right)$$

$$= \partial_2 f^{\cancel{*}}(s,x+t-s) - \partial_2 f^{*}(s,x-t+s) \cancel{\;\;}$$

Hence $w_{tt}(t,x) = f(t,x) + \frac{1}{2}\int_0^t \partial_2 f(s,x+t-s) - \partial_2 f(s,x-t+s)\,ds$

So $w_{tt} - w_{xx} = f(t,x)$ //

② $f \in C^2((0,\infty)\times\mathbb{R})$ $g \in C^2(\mathbb{R})$ $h \in C^1(\mathbb{R})$

$u_{tt} - u_{xx} = f$ in $(0,\infty)\times\mathbb{R}$

$u = g$, $u_t = h$ on $\{t=0\}\times\mathbb{R}$

(a) Show: If $f = g = 0$, soln given by

$u_{\not{t}}(t,x) = \int_{\mathbb{R}} K(t, x-y)\, h(y)\, dy$ $K(t,x) = \frac{1}{2} H(|t| - |x|)\, \mathrm{sgn}(t)$

$H = \chi_{[0,\infty)}$

By d'Alembert,

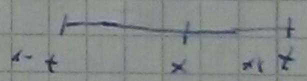
$u_{\not{t}}(t,x) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$

$= \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$

$= \cancel{\tfrac{1}{2}} \int_{\mathbb{R}} \frac{1}{2} \chi_{(x-t,x+t)}(y)\, h(y)\, dy \quad \times\, \mathrm{sgn}(t)$

Need to show $\underbrace{\chi_{[x-t,x+t]}(y)}_{\alpha(y)} = \underbrace{H(|t| - |\overset{x-y}{x}|)\, \mathrm{sgn}(t)(y)}_{\beta(y)}$ $\forall y \in \mathbb{R}$.

If LHS $\cancel{\alpha(y)} = 1$ iff $\cancel{x}\ y \in (x-t, x+t)$ iff



iff $y \in (\cancel{t}\ x-|t|, x+|t|)$

iff $x-y \in [-|t|, |t|]$

iff $|t| \geq |x-y|$.

ie $\chi_{[x-t,x+t]}(y) = \cancel{x}\ 1$ iff $|t| - |x-y| \geq 0$.

so $\chi_{[x-t,x+t]}(y) = K(t, \cancel{x})\ H(|t| - |x \overset{-y}{}|)$

(b) Suppose $g = h = 0$. (f nonzero)

Duhamel's principle:

define $u = u(t, x; s)$ to be a sol of

$$u_{tt}(t,x;s) - u_{xx}(t,x;s) = 0 \qquad (t,x) \in (s,\infty) \times \mathbb{R}$$
$$u(s,x;s) = 0, \; u_t(s,x;s) = f(s,x) \qquad \text{on } \{t=s\} \times \mathbb{R}.$$

Then $u(t,x) = \int_0^t u(t, x; s)\, ds$

is a sol of inhom wave eqn.

By (a), $u(t,x;s) = \int_{\mathbb{R}} K(t-s, x-y) f(s,y)\, dy$

Here sol is:

$$u(t,x) := \int_0^t \int_{\mathbb{R}} K(t-s, x-y) f(s,y)\, dy.$$

Need
$$\begin{cases} u^{(t-s,x)}_{tt} - u^{(t-s)}_{xx} = 0 & (0,\infty) \times \mathbb{R} \\ u(0,x) = 0, \; u_t(0,s) = f(s,x) & \text{on } \{0\} \times \mathbb{R} \end{cases}$$

(b) (Neater) Duhamel's principle:

For $s > 0$, Define $u^{(s)} = u(t,x,s)$ to be a solution of

$$(*)\begin{cases} u^{(s)}_{tt}(\cdot; s) - u^{(s)}_{xx}(\cdot; s) = 0 & \text{in } (s,\infty) \times \mathbb{R} \\ u^{(s)}(s,x;s) = 0, \; u^{(s)}_t(s,x;s) = f(s,x) & \text{on } \{t=s\} \times \mathbb{R}. \end{cases}$$

Then solutions to $(*^2)\begin{cases} u_{tt} - u_{xx} = f & (0,\infty) \times \mathbb{R} \\ u = 0, \; u_t = 0 & \text{on } \{t=0\} \times \mathbb{R} \end{cases}$

is given by $u(t,x) = \int_0^t u(t,x;s)\, ds$

~~For $s>0$ let $v^s(t,x) := u(t-s,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$.~~

Fix $s > 0$: Let $f^s(x) := f(s, x)$

Then by (a) $\begin{cases} v^s_{tt} - v^s_{xx} = 0 & (0,\infty) \times \mathbb{R} \\ v^s = 0, \; v^s_t = f^s(x) & \{t=0\} \times \mathbb{R} \end{cases}$

Given by $v^s(t,x) = \int_{\mathbb{R}} K(t, x-y) f^s(y)\, dy \qquad (t,x) \in (0,\infty) \times \mathbb{R}$

③ $u_{xy}(x,y) = 0$. Write $v(x,y) = u_y(x,y)$

$\Rightarrow \quad \frac{\partial}{\partial x}(v(x,y)) = 0.$

So $v(x,y) = \zeta(y) = u_y(x,y)$

$$u(x,y) = \int_0^{y} \zeta(t)\,dt + \xi(x)$$

$$= \tilde{\zeta}(y) + \xi(x).$$

Conversely if $u(x,y) = \xi(x) + \zeta(y)$

$u_{xy} = 0.$

④ u soln of oscillating string prob on $\mathbb{R}_+$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } (0,\infty)\times\mathbb{R}_+ \\ u = g, \ u_t = h & \text{on } \{t=0\}\times\mathbb{R}_+. \end{cases} \qquad u = 0 \quad (0,\infty)\times\{x=0\}$$

d'Alembert for half line:

$$u(t,x) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \geq t \geq 0 \\ \frac{1}{2}[g(x+t) - g(\overset{t-x}{\cancel{x-t}})] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & t \geq x \geq 0. \end{cases}$$

Then (see qu 1): For $u \in C^2$, need then u + derivs cont at $x = t$.

$$u_x(t,x) = \begin{cases} \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t) - h(x-t)] & x \geq t \\ \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t) + h(\overset{-x+t}{\cancel{x-t+x}})] & t \leq x \end{cases}$$

$$u_{xx}(t,x) = \begin{cases} \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(x-t)] & x \geq t \\ \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(t - x)] & t \leq x \end{cases}$$

u_{xx} to be cont at $x=t$. Need $\lim_{x-t\to 0} -h(x-t) = \lim_{x-t\to 0} h(-t+x)$

ie $-h(0) = h(0)$. $h(0) = 0$.

Proceed similarly to try u_t, u_{tt} u_{tx} to show need $g(0) = g''(0) = 0$

(3) Suppose $u \in C^2(\mathbb{R}^2)$ and $u_{xy} = 0$ on $\mathbb{R}^2$.

Fix $x_0 \in \mathbb{R}$ and consider $v(y) := u_x(x, y)$, $y \in \mathbb{R}$. $v \in C^1(\mathbb{R})$ $(u \in C^2(\mathbb{R}^2))$

Then $v'(y) = u_{xy}(x,y) = 0 \quad \forall y \in \mathbb{R}$.

So $v'(y) = C_x$ constant depending on x.

True for all $x \in \mathbb{R}$. Write $f(x) := C_x$, $x \in \mathbb{R}$.

So we have $u_x(x,y) = f(x) \quad \forall (x,y) \in \mathbb{R}^2$.

Then, for fixed $y \in \mathbb{R}$ let $w(x) := u(x,y)$, $x \in \mathbb{R}$.

$u \in C^2$. Then $w'(x) = u_x(x,y) = f(x)$

So $w(x) = \underbrace{\int_0^x f(t)\,dt}_{= \tilde{f}(x),\ \text{say}} + C_y$ True for all $y \in \mathbb{R}$.

So

$$u(x,y) = \tilde{f}(x) + g(y)$$

Conversely, if $u(x,y) = \xi(x) + \zeta(y)$ clearly $u_{xy} = 0$.

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(2)(b) (cont) Let $u(t,x;s) = v^s(t-s, x)$ $(t,x) \in (s,\infty) \times \mathbb{R}$.

Then $u(t,x;s)$ solves $(*)$

So solution to $(*^2)$ is

$$u(t,x) = \int_0^t v^s(t-s, x)\,ds$$

$$= \int_0^t \int_{\mathbb{R}} K(t-s, x-y) f(s,y)\,ds \quad //$$